Constrained quantization in algebraic field theory

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Quantization relates Poisson algebras to C^*-algebras. The analysis of local gauge symmetries in algebraic quantum field theory is approached through the quantization of classical gauge theories, regarded as constrained dynamical systems. This approach is based on the analogy between symplectic reduction and Rieffel induction on the classical and on the quantum side, respectively. Thus one is led to a description of e.g. θ-angles and gauge anomalies in the algebraic setting.

1 Quantization of observables

The common structure of the algebra of observables of a classical or a quantum system is that of a Jordan-Lie algebra. This is a real vector space $\mathfrak{A}$ equipped with two bilinear maps $\circ$ and $\{\ ,\ \}$, where $\circ$ is symmetric and $\{\ ,\ \}$ is anti-symmetric. For each $A \in \mathfrak{A}$ the map $B \to \{A, B\}$ must be a derivation of $\circ$ as well as of $\{\ ,\ \}$ (it follows that the latter is a Lie bracket), and the identity $(A \circ B) \circ C - A \circ (B \circ C) = k\{\{A, C\}, B\}$ must hold for some $k \in \mathbb{R}$. A Poisson algebra is a Jordan-Lie algebra in which $k = 0$ [1, 2].

The main sources of Poisson algebras are Poisson manifolds, which are smooth manifolds P equipped with a Poisson bracket $\{\ ,\ \}$ on $\mathfrak{A} = C^\infty(P)$. The product $f \circ g$ is then simply given by pointwise multiplication fg. In quantization theory one often works with a sup-dense subalgebra $\tilde{\mathfrak{A}}^0$ of $C_0(P)$, rather than with $C^\infty(P)$ [2].

Jordan-Lie algebras with $k > 0$ are obtained by taking $\mathfrak{A}$ to be the self-adjoint part of a C^*-algebra $\mathfrak{A}_\mathbb{C}$. One then takes $A \circ B = \frac{1}{2}(AB + BA)$ and $\{A, B\} = i(AB - BA)/\hbar$ for some $\hbar \in \mathbb{R}\backslash\{0\}$; note that these maps indeed preserve self-adjointness (unlike the associative product in $\mathfrak{A}_\mathbb{C}$). One then has $k = \hbar^2/4$. (Similarly, to get $k < 0$ one takes $\mathfrak{A}$ to be an R^*-algebra and omits the i.) A Jordan-Lie algebra $\mathfrak{A}$ with $k > 0$ is the the self-adjoint part of a C^*-algebra iff $\mathfrak{A}$ is a so-called JB-algebra in $\circ$, in other words, iff it is a Banach algebra in which the inequality $\| A \|^2 \leq \| A^2 + B^2 \|$ holds (here $A^2 = A \circ A$).

These considerations inspire the definition of a strict quantization of a Poisson algebra $\mathfrak{A}^0$ as a family of C^*-algebras $\{\mathfrak{A}^\hbar_\mathbb{C}\}_{\hbar \in I}$, where $I\backslash\{0\} \subseteq \mathbb{R}$ has 0 as an accumulation point, and for each $\hbar$ one has a linear map $Q_\hbar : \tilde{\mathfrak{A}}^0 \to \mathfrak{A}^\hbar$ whose image is dense. One naturally requires that for all $f \in \tilde{\mathfrak{A}}^0$ the function $\hbar \to \| Q_\hbar(f) \|$ is continuous on I (in particular, $\lim_{\hbar\to 0} \| Q_\hbar(f) \| = \| f \|_\infty$), that $\lim_{\hbar\to 0} \| \{Q_\hbar(f), Q_\hbar(g)\}_\hbar - Q_\hbar(\{f, g\}) \| = 0$, and finally that $\lim_{\hbar\to 0} \| Q_\hbar(f) \circ Q_\hbar(g) - Q_\hbar(f \cdot g) \| = 0$. In the present form the first two conditions were proposed by Rieffel [3]; a heuristic version of the last two goes back to Dirac and von Neumann, respectively. 'Deformation' quantization 'deforms' the symmetric product on $\tilde{\mathfrak{A}}^0$; this is equivalent to the present approach iff $Q_\hbar(\tilde{\mathfrak{A}}^0)$ is closed under multiplication and $Q_\hbar(f) = 0$ implies $f = 0$. Our approach is particularly suitable to describe what currently is called Berezin-Toeplitz quantization.

2 Quantization of pure states

One may look at quantization from the perspective of pure states rather than algebras. This requires an understanding of the extent to which the properties of a C^*-algebra $\mathfrak{A}_\mathbb{C}$ are encoded in its pure state space $\mathcal{P}(\mathfrak{A})$. It turns out that $\mathcal{P}(\mathfrak{A})$ should be seen as a Poisson space with a transition probability; this is a generalization of a Poisson manifold which is not a manifold itself (but rather a uniform space in the sense of point-set topology), but nonetheless is foliated by symplectic leaves (in the smooth case these are the "irreducible" subspaces of a Poisson manifold, which are somewhat comparable with the primitive ideals of an algebra). The symplectic leaves of the pure state space of a C^*-algebra are projective Hilbert spaces equipped with the well-known Fubini-Study symplectic form.

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A transition probability on a pure state space $\mathcal{P}$ is a function $p : \mathcal{P} \times \mathcal{P} \to [0,1]$ with appropriate properties, such as $p(\rho,\sigma) = 1$ iff $\rho = \sigma$. On a classical pure state space one has $p(\rho,\sigma) = \delta_{\rho\sigma}$, whereas on a C^*-algebra the function p is defined through the inner product on the Hilbert spaces defining the symplectic leaves (which are orthogonal to each other) in the well-known way. Roughly speaking, the Poisson structure on $\mathcal{P}(\mathfrak{A})$ encodes the commutator on $\mathfrak{A}$, whereas the transition probability encodes the Jordan product $\circ$; in conjunction these define the associative product on $\mathfrak{A}_{\mathbb{C}}$. The norm on $\mathfrak{A}$ is related to the uniform structure on $\mathcal{P}(\mathfrak{A})$. See [1, 2].

The 'dual' picture of quantization is now as follows [2]: one has a family of injections $q_\hbar$ of a Poisson manifold P into the pure state space $\mathcal{P}(\mathfrak{A}^\hbar)$ of a C^*-algebra $\mathfrak{A}^\hbar_{\mathbb{C}}$, such that $\lim_{\hbar\to 0} p(q_\hbar(\rho), q_\hbar(\sigma)) = \delta_{\rho\sigma}$. In other words, the quantum-mechanical transition probabilities should converge to the classical ones. Moreover, the $q_\hbar$ should relate the Poisson structure on P to that on each $\mathcal{P}(\mathfrak{A}^\hbar)$ as appropriate.

Under suitable conditions the quantization $q_\hbar$ of a (locally compact) symplectic manifold $P = S$ leads to a strict quantization $Q_\hbar$ of $\tilde{\mathfrak{A}}^0 = C_c^\infty(S)$ into $\mathfrak{A}^\hbar$ [2]: one defines $Q_\hbar(f) \in \mathfrak{A}^\hbar$ as the operator whose expectation value in a pure state ψ on $\mathfrak{A}^\hbar$ equals

$$\psi(Q_\hbar(f)) = \int_S d\mu^\hbar(\sigma)\, p(q_\hbar(\sigma), \psi) f(\sigma).$$

Here $\mu^\hbar$ is a measure on S which is usually proportional to the Liouville measure μ_L; the proportionalilty constant blows up as $\hbar \to 0$. In this case $Q_\hbar$ is a completely positive map, and the Stinespring theorem assigns a natural role to the Hilbert space $L^2(S, \mu_L)$ that always pops up in geometric quantization and in the Koopman approach to classical mechanics. For example, Berezin-Toeplitz quantization is of this sort.

It should be emphasized that the approach advertised here is not a technique for obtaining quantum mechanics from classical methods (such as geometric quantization); it is rather the study of the correspondence between intrinsically defined classical and quantum theories. Even if one cannot relate a given Poisson algebra (or Poisson manifold) to a C^*-algebra (or its pure state space) by quantization, one may seek constructions and theorems in both settings that are 'morally' equivalent. This idea plays an essential role in what follows.

3 Classical reduction

Classical gauge theories may be described in terms of symplectic reduction [4], in that the phase space S^p of physical degrees of freedom may be obtained from its unphysical counterpart S as an infinite-dimensional Marsden-Weinstein quotient [5] with respect to the gauge group $\mathcal{G}$ (whose Lie algebra is denoted by $\mathfrak{g}$). This reduction procedure may be carried out covariantly; for simplicity we work in the temporal gauge $A_0 = 0$. In that case S is an appropriate space of spatial gauge fields A and their conjugate momenta E. Let $J : S \to \mathfrak{g}^*$ be the momentum map defined by the action of $\mathcal{G}$ on S. The constraint $J = 0$ is Gauss' law, so that the reduced phase space $J^{-1}(0)/\mathcal{G}$, in which gauge-equivalent field configurations are identified, indeed equals S^p.

From the perspective of the algebra of observables the reduction procedure looks as follows: one defines the Poisson algebra of weak observables $\mathfrak{A}^p_w$ as the set of $\mathcal{G}$-invariant functions f in $C^\infty(S)$ (where $f \in \mathfrak{A}^p_w$), and subsequently constructs the physical observable $\pi^p(f)$ in $C^\infty(S^p)$ as the restriction of f to $J^{-1}(0)$. The Poisson algebra of classical physical observables is $\mathfrak{A} = \pi^p(\mathfrak{A}^p_w)$, which is evidently a quotient of $\mathfrak{A}^p_w$.

As a simple illustration we consider Yang-Mills theory on a circle $\mathbb{T}$ with structure group G (whose Lie algebra we denote by $\mathbf{g}$). In that case the gauge group $\mathcal{G}$ is a loop group on G, and the appropriate choice is to define $\mathcal{G}$ to consist of the Sobolev space of all continuous loops with finite energy. In other words, $\mathcal{G} = H_1(\mathbb{T}, G)$. This forces the space $\mathcal{A}$ of all connections to be the real Hilbert space $L^2(\mathbb{T}, \mathbf{g})$, so that $S = T^*\mathcal{A} \simeq L^2(\mathbb{T}, \mathbf{g}_{\mathbb{C}})$. One can solve Gauss' law on the circle in terms of Wilson loops, and the reduced phase space comes out to be $T^*(G/Ad(G))$. This is a finite-dimensional space! See [6] for details and references to the literature on this fashionable topic.

To see how this is to be quantized it turns out to be helpful to generalize the Marsden-Weinstein reduction procedure [7]. Instead of $\mathfrak{g}^*$ we consider an arbitrary Poisson manifold P, with realization S,

i.e., we suppose an (anti) Poisson morphism $J : S \rightarrow P$ is given (the 'generalized momentum map'), with S symplectic. Let a second realization $\rho : S_\rho \rightarrow P$ be given. We then form the fiber product $C = S *_P S_\rho = \{(x, y) \in S \times S_\rho | J(x) = J_\rho(y)\}$. The reduced space is then defined as $S^\rho = (S *_P S_\rho)/\mathcal{F}_0$, where $\mathcal{F}_0$ is the null foliation of the induced pre-symplectic form on $S *_P S_\rho$.

When $\mathcal{G}$ is connected and J is the usual momentum map, this construction specializes to Marsden-Weinstein reduction if we choose $S_\rho = \{0\}$ (or, more generally, as an arbitrary coadjoint orbit in $\mathfrak{g}^*$) and ρ as the inclusion map. One then has $\pi^0 = \pi^p$ as defined earlier. The case where $\mathcal{G}$ is disconnected will be discussed later on.

Back to the general case, let $\mathfrak{A}_w$ be the Poisson subalgebra of $C^\infty(S)$ consisting of all functions f with the property that for all $g \in C^\infty(P)$ the Poisson bracket $\{J^*g, f\}$ vanishes at all points of S which are relevant to C. We can define an 'induced representation' π^ρ of $\mathfrak{A}_w$ on S^ρ by $\pi^\rho(f)([x, y]) = f(x)$, where $[x, y] \in S^\rho$ is the image of $(x, y) \in C$ under the canonical projection from C to $C/\mathcal{F}_0$. By our definition of $\mathfrak{A}_w$ this is well-defined, and the image $\pi^\rho(\mathfrak{A}_w)$ is a quotient of $\mathfrak{A}_w$.

4 Quantum induction

Traditional constrained quantization (Dirac) quantizes S into a Hilbert space $\mathcal{H}$, and subsequently attempts to define the Hilbert space of physical states as a subspace of $\mathcal{H}$ on which the quantized constraints hold. This leads to endless problems. Instead, we propose a quantum version of the classical reduction procedure in the previous section as a whole. The central idea is to 'quantize' the unconstrained system in such a way that the presence of constraints is taken into account, without assigning a specific value to them.

The mathematical implementation of this idea uses the concept of a (Hilbert) C^*-module. This is a complex linear space L equipped with a right-action of a C^*-algebra $\mathfrak{B}$, as well as a $\mathfrak{B}$-valued inner product $\langle\ ,\ \rangle_{\mathfrak{B}}$ which is positive (i.e., $\langle\psi, \psi\rangle_{\mathfrak{B}} \geq 0$ in $\mathfrak{B}$) and $\mathfrak{B}$-equivariant in that $\langle\varphi, \psi B\rangle_{\mathfrak{B}} = \langle\varphi, \psi\rangle_{\mathfrak{B}} B$. Also, L must be closed in the norm $\| \psi \|^2 = \| \langle\psi, \psi\rangle_{\mathfrak{B}} \|$. Assuming that $\mathfrak{B}$ is the quantization of a suitable subalgebra of $C^\infty(P)$ (cf. section 1), the right-action of $\mathfrak{B}$ on L is to be thought of as the quantization of the pull-back $J^* : C^\infty(P) \rightarrow C^\infty(S)$ of the generalized momentum map $J : S \rightarrow P$, whereas $\langle\ ,\ \rangle_{\mathfrak{B}}$ is the quantization of J itself [7].

The quantum analogue of symplectic reduction is then given by Rieffel induction [8]. This procedure starts from a represenation $\pi_\rho(\mathfrak{B})$ on a Hilbert space $\mathcal{H}_\rho$ (with inner product $(\ ,\ )_\rho$), which is the quantum counterpart of the pull-back of the realization $\rho : S_\rho \rightarrow P$ (choosing π_ρ assigns a value to the constraints in the appropriate quantum-mechanical sense). This leads to a sesquilinear form $(\ ,\ )_0$ on $L \otimes \mathcal{H}_\rho$, defined by linear extension of $(\psi \otimes v, \varphi \otimes w)_0 = (\pi_\rho(\langle\varphi, \psi\rangle_{\mathfrak{B}})v, w)_\rho$. This form is positive semi-definite, with null space $\mathcal{N}$; the induced space $\mathcal{H}^\rho$ is simply defined as the completion of $L \otimes \mathcal{H}_\rho/\mathcal{N}$ in the inner product inherited from $(\ ,\ )_0$.

The step of quotienting by $\mathcal{N}$ is evidently the quantum analogue of the passage from the classical constraint hypersurface C to the physical phase space $C/\mathcal{F}_0$. However, the set C on which the classical constraints hold has no quantum counterpart; in quantum mechanics it is not necessary to solve the constraints. This is a crucial difference between the present approach and traditional ones such as that of Dirac (who, to give due credit, did recognize that only one of the two steps of classical reduction needed to be quantized; unfortunately, he picked the wrong one).

The algebra $\mathfrak{A}_w$ of weak quantum observables of the constrained system consists of those operators A on L which are self-adjoint with respect to $(\ ,\ )_0$, i.e., $(A\Psi, \Phi)_0 = (\Psi, A\Phi)_0$ for all $\Psi, \Phi \in L \otimes \mathcal{H}_\rho$ (here A is identified with $A \otimes \mathbb{I}$). For such operators $A\mathcal{N} \subseteq \mathcal{N}$, so that the quotient action $\pi^\rho(A)$ on $L \otimes \mathcal{H}_\rho/\mathcal{N}$ is well-defined. This 'induced representation' may be extended to $\mathcal{H}^\rho$ when a suitable boundedness assumption is satisfied. The quantum algebra of physical observables $\pi^\rho(\mathfrak{A}_w)$ is then a quotient of the algebra $\mathfrak{A}_w$ of weak observables.

The idea that Rieffel induction is the quantum analogue of symplectic reduction is supported by the fact that the main theorems on induced representations (the theorem on induction in stages and the imprimitivity theorem [8]) have analogues for symplectic reduction, whose proofs may almost be copied from the 'quantum' case [7, 2] (also see [9] for a different approach to the symplectic imprimitivity theorem).

5 Gauge theory on a circle

The classical Marsden-Weinstein quotient $S^0 = J^{-1}(0)/\mathcal{G}$ with finite-dimensional $\mathcal{G}$ may be quantized according to the above scheme. One starts by quantizing the unconstrained phase space S with its given $\mathcal{G}$-action by a Hilbert space $\mathcal{H}$ carrying a unitary representation U of $\mathcal{G}$, and subsequently tries to find a dense subspace $L_0 \subseteq \mathcal{H}$ such that the function $x \to (U(x)\Psi, \Phi)$ is in $L^1(G)$ for all $\Psi, \Phi \in L_0$. When G is compact one may choose $L_0 = \mathcal{H}$. With $\mathfrak{B} = C^*(G)$ the $\mathfrak{B}$-valued inner product on L_0 is defined by $\langle \Psi, \Phi \rangle_{C^*(G)} = (U(x)\Phi, \Psi)$. Under suitable conditions this may be closed into a C^*-module L; note that in general $L \neq \mathcal{H}$. There is a bijective correspondence between unitary representations U_ρ of G and representations π_ρ of the group algebra $C^*(G)$, given (initially on $L^1(G)$) by $\pi^\rho(f) = \int_G dx f(x) U_\rho(x)$. Hence the form $(\ ,\ )_0$ on $L \otimes \mathcal{H}_\rho$ comes out as the group average

$$(\Psi, \Phi)_0 = \int_{\mathcal{G}} dx\, (U(x) \otimes U_\rho(x)\Psi, \Phi).$$

One may start the induction procedure from this expression, forgetting $C^*(G)$ and the derivation of the formula. This philosophy is necessary in applying the technique to gauge theories, where the gauge group $\mathcal{G}$ is not locally compact and fails to possess a Haar measure dx, so that $C^*(G)$ is not defined. We illustrate this for a gauge theory on a circle [6], as discussed in section 3 (also see [10] for abelian gauge theories on $\mathbb{R}^3$).

For the Hilbert space $\mathcal{H}$ of the unconstrained system we take the bosonic Fock space $\exp(S)$ over the classical phase space $S = L^2(\mathbb{T}, \mathfrak{g}_{\mathbb{C}})$, and for $U(\mathcal{G})$ we take the well-known 'energy representation' [11]. The domain L_0 is the linear span of the exponential (or 'coherent') vectors in $\exp(S)$; these are of the form $\sqrt{\exp}(A) = \sum_{n=0}^{\infty} \otimes^n A/\sqrt{n!}$. It turns out that, heuristically speaking, the would-be Haar measure on $\mathcal{G}$ combines with a Gaussian factor in the matrix element of $U(x)$ to form the Wiener measure μ_W (cf. its emergence in the Feynman-Kac formula). Since $\mathcal{G}$ has μ_W-measure zero, the domain of integration must be extended from $\mathcal{G}$ to its sup-closure $C(\mathbb{T}, G)$ (the fact that U cannot be extended beyond $\mathcal{G}$ is inconsequential, since $(\Psi, \Phi)_0$ remains well-defined).

Since the physical phase space of the classical theory is obtained by reduction from 0, it seems natural to define its quantum counterpart by inducing from the trivial representation $U_0(\mathcal{G}) = 1$ on $\mathcal{H}_0 = \mathbb{C}$ (so that $L \otimes \mathcal{H}_\rho \simeq L$). The resulting Wiener integral over $C(\mathbb{T}, G)$ can be performed exactly, so that the explicit form of the induced space $\mathcal{H}^0$ can be derived. This is most easily done by guessing what $\mathcal{H}^0$ should be (in the present case heuristic approaches suggest that $\mathcal{H}^0_{\text{guess}} = L^2(G/Ad(G))$), and then confirming this guess by constructing a map $V : L \to \mathcal{H}^0_{\text{guess}}$ satisfying $(V\Psi, V\Phi) = (\Psi, \Phi)_0$. For when $\mathcal{H}^0_{\text{guess}}$ indeed equals $\mathcal{H}^0$, the map V is simply the canonical projection, in terms of which $\pi^0(A)V = VA$.

Interestingly [12], if one reduces by the group of based gauge transformations, so that $S^p = T^*G$ and $\mathcal{H}^0 = L^2(G)$, the image of an exponential vector $\sqrt{\exp}(A)$ is always a coherent state in $L^2(G)$ in the sense of Hall [13]. This fact allows for an efficient computation of the induced representation π^0.

6 θ-angles and anomalies

The preceding section may be summarized by saying that the algebra of observables is constructed by inducing from the trivial representation of the gauge group. This was justified by the corresponding construction of the classical phase space by reduction from the zero level of the momentum map. In case that the gauge group $\mathcal{G}$ is disconnected, however, there is no good reason why one should not induce from a unitary representation $U_\theta = \tilde{U}_\theta \circ \tau$, where $\tilde{U}_\theta$ is a unitary representation of $\mathcal{G}/\mathcal{G}_0$ and $\tau : \mathcal{G} \to \mathcal{G}/\mathcal{G}_0$ is the canonical projection (here $\mathcal{G}_0$ is the component of $\mathcal{G}$ containing the unit element).

The origin of this freedom lies in the fact that the quotient S/D of a symplectic manifold S by a discrete group D is symplectic, unlike the quotient by a Lie group of dimension > 0. Hence there is no momentum map J and no "$0''$ in $J^{-1}(0)$. In other words, there is no constraint in the classical theory. To apply this argument to gauge theories one reduces and induces in stages, firstly with $\mathcal{G}_0$ and secondly with the discrete group $\mathcal{G}/\mathcal{G}_0$. We conclude that in the present approach (generalized) θ-angles emerge from the freedom to induce from $\tilde{U}_\theta(\mathcal{G})$ rather than from the trivial representation.

Possible gauge anomalies appear when U is merely a projective representation of $\mathcal{G}$.

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